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THE DESIGN OF ORTHOTROPIC MATERIALS FOR STRESS OPTIMIZATION

ARTHUR W. LEISSA
Professor of Engineering Mechanics, Ohio State University, Columbus, OH 43210, U.S.A.

and

MILTON VAGINS
Manager, Structures and Mechanics, Battelle, Northwest Richland, Washington, U.S.A.

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Abstract—The availability of materials such as fibrous composites now makes it feasible to approach problems of the theory of elasticity from a different perspective. Rather than determining the stresses within a homogeneous body subjected to known loads, it may be possible to design a nonhomogeneous material to yield an optimum internal stress field. The approach is demonstrated for two dimensional problems in polar coordinates, with detailed results given for the problems of the pressurized, thick-walled cylinder and the rotating disk subjected to centrifugal body forces.

NOTATION

A, B constants used in expressions for stresses
a, b internal and external radii, respectively
c_{ij} ratios of elastic coefficients (see eqns 6) for a dependently homogeneous material
E modulus of elasticity
E_r, E_θ orthotropic moduli
E_θ^0, E_θ^i circumferential moduli on the outer and inner boundaries, respectively
$G_{r\theta}$ shear modulus
k b/a
p_i, p_0 internal and external boundary pressures, respectively
R body force (per unit volume) in the radial direction
r radial coordinate
V potential function
α_{ij} elastic coefficients (constants for a homogeneous material)
$\gamma_{r\theta}$ shear strain in the $r\theta$-plane
ϵ_r, ϵ_θ normal strains in the r- and θ-directions, respectively
Θ body force (per unit volume) in the θ-direction
θ circumferential coordinate
ν Poisson's ratio
$\nu_{r\theta}$, $\nu_{\theta r}$ orthotropic Poisson's ratios
ρ mass density (mass/unit volume)
σ_r, σ_θ normal stresses in the r- and θ-directions, respectively
$\tau_{r\theta}$ shear stress in the $r\theta$-plane
ϕ Airy stress function
ψ auxiliary stress function
ω angular velocity of a rotating disk (radians/unit time)
∇^2 Laplacian differential operator
∇^4 biharmonic differential operator ($= \nabla^2\nabla^2$)

1. INTRODUCTION

The classical problem of the theory of elasticity may be stated as: Given a body having specified shape which is subjected to specified boundary conditions of stress (and/or displacement), *and for which the material properties at every point are prescribed, determine the internal stress (and/or displacement) field.* The literature is replete with countless works pertaining to this problem. Nearly all of the work assumes that the material is homogeneous, although a few references can be found dealing with nonhomogeneous materials.

A converse problem may be stated as: Given a body having specified shape which is subjected to specified boundary conditions of stress (and/or displacement), *and for which the internal stress (and/or displacement) field are prescribed, determine the required material properties at every point.* Virtually nothing has been written about this problem. It requires assuming, in general, that the material is nonhomogeneous.

For example, consider a flat panel subjected to a uniaxial tensile stress caused by external loading applied to its boundaries. Let a small circular hole be drilled into the panel. The classical solution of Kirsch[1] for an isotropic, homogeneous material is widely known (see Timoshenko and Goodier[2], p. 90) and reveals that the presence of the hole increases the maximum stresses present by a factor of three. If the panel were composed of an *orthotropic* material, then no simple, closed form solution is available to describe the stress concentration in the vicinity of the hole. Nevertheless, the stress field could be calculated by one of several analytical methods to any degree of accuracy desired, and the magnitude of the stress concentration would be seen to depend upon the several independent elastic constants of the orthotropic material to be used.

One can imagine further that a fibrous composite is employed (e.g. glass or boron fibers imbedded in an epoxy matrix), which can be modeled on the macroscopic scale as an orthotropic material. By varying the relative spacing of the fibers the effective elastic constants can be varied; i.e. the material has controllable nonhomogeneity in the macroscopic sense. Furthermore, by varying the fiber orientation and spacing in the vicinity of the hole, it seems clear enough that stress concentration can be reduced, and perhaps even eliminated altogether.

As another example, consider the two-dimensional plane strain problem of the thick-walled, hollow cylinder subjected to internal pressure. For the case of a homogeneous, isotropic material the well-known solution to this problem (see Lamé[3] and Timoshenko and Goodier[2], p. 69) indicates that stress concentration always exists at the inner boundary of the cylinder. On the other hand, the present authors have shown in a recent paper[4] that stress concentration can be altogether eliminated in this problem by proper variation of the modulus of elasticity (E) in an isotropic material. That is, the material is designed to yield a desired stress field within the body for the given loading.

In the present work the equations of elasticity for a nonhomogeneous, orthotropic material are expressed for convenience in polar coordinates, and special attention is given to the case of a dependently nonhomogeneous, orthotropic material. The solution of the equations is demonstrated for two problems: (1) the pressurized cylinder and (2) the rotating disk. Variations in elastic moduli are found which permit the elimination of stress concentration for these two problems. Further possible extensions of the work are also discussed.

2. NONHOMOGENEOUS ORTHOTROPIC ANALYSIS IN PLANE POLAR COORDINATES

Limiting oneself to two dimensional problems of elastic deformation the equation of compatibility to be satisfied is

$$\frac{\partial}{\partial r}\left(r\frac{\partial \gamma_{r\theta}}{\partial \theta} - r^2\frac{\partial \epsilon_\theta}{\partial r}\right) + r\frac{\partial \epsilon_r}{\partial r} - \frac{\partial^2 \epsilon_r}{\partial \theta^2} = 0. \tag{1}$$

It is also necessary to satisfy the equations of equilibrium. This is conveniently done by defining the stresses in terms of an Airy stress function, $\phi = \phi(r, \theta)$, as

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + V\\ \sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} + V\\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)\end{aligned} \tag{2}$$

where the potential function, $V = V(r, \theta)$, has been added to provide for body force intensities derivable from V by

$$R = -\frac{\partial V}{\partial r}, \qquad \Theta = -\frac{1}{r}\frac{\partial V}{\partial \theta}. \tag{3}$$

The strains are related to the stresses by the orthotropic relationships

$$\epsilon_r = \alpha_{11}\sigma_r + \alpha_{12}\sigma_\theta = \frac{1}{E_r}(\sigma_r - \nu_{r\theta}\sigma_\theta)$$

$$\epsilon_\theta = \alpha_{12}\sigma_r + \alpha_{22}\sigma_\theta = \frac{1}{E_\theta}(-\nu_{\theta r}\sigma_r + \sigma_\theta) \quad (4)$$

$$\gamma_{r\theta} = \alpha_{33}\tau_{r\theta} = \frac{1}{G_{r\theta}}\tau_{r\theta}$$

where the elastic coefficients for the nonhomogeneous material are varying with the spatial coordinates; i.e. $\alpha_{ij} = \alpha_{ij}(r, \theta)$, etc. The inclusion of thermal effects in the problem is simply a matter of adding the proper orthotropic terms to eqns (4).

The usual procedure in solving a problem where the boundary conditions are given in terms of stresses is to substitute eqns (2) and (4) into (1), thereby expressing the problem in terms of the unknown function, ϕ. For the case of a homogeneous, isotropic material this procedure results in the well-known equation

$$\nabla^4\phi + (1-\nu)\nabla^2 V = 0 \quad (5)$$

where the presence of V requires the addition of a particular solution to the biharmonic complementary solution. For a homogeneous, *orthotropic* material eqn (5) becomes generalized to a form which is more complicated, but it remains a differential equation having constant coefficients. For a *nonhomogeneous*, orthotropic material the resulting equation would be considerably longer and more complicated (it would require approximately one page of the journal to write). Moreover, the equation would have variable coefficients.

The complexity of the nonhomogeneous, orthotropic problem can be reduced considerably by employing a simplifying assumption similar to the one used by Bert[5] for the analysis of orthotropic circular disks having variable thickness. That is, it will be assumed that, although the four elastic coefficients α_{11}, α_{12}, α_{22} and α_{33} each will vary with r and θ, they will vary *in the same manner.* Taking α_{22} as the basic elastic coefficient, the others can then be expressed in terms of α_{22} by a set of *linear* relationships:

$$\alpha_{11}(r, \theta) = c_{11}\alpha_{22}(r, \theta)$$

$$\alpha_{12}(r, \theta) = c_{12}\alpha_{22}(r, \theta)$$

$$\alpha_{33}(r, \theta) = c_{33}\alpha_{22}(r, \theta) \quad (6)$$

where the c_{ij} are *constants* determining the relative magnitudes of the elastic coefficients throughout the nonhomogeneous body. That the elastic coefficients should vary from point to point in the same manner is a reasonable approximation for most materials which are nonhomogeneous in the macroscopic sense. Let us say that an orthotropic material restrained according to eqns (6) is "dependently nonhomogeneous".

The problem of nonhomogeneous, orthotropic elasticity in plane polar coordinates thus requires the satisfaction of eqn (1), with the strains expressed in terms of the stresses by eqns (4), further constrained in the case of a dependently nonhomogeneous, orthotropic material by eqns (6), and the stresses expressed in terms of ϕ by eqns (2).

3. THE THICK-WALLED, PRESSURIZED CYLINDER

As a first example of the design of materials to optimize stresses, consider the thick-walled cylinder subjected to internal or external pressure, or both. Figure 1 depicts a cylinder having internal and external radii of a and b, respectively, subjected to the internal pressure, p_i. The case of external pressure will be considered later.

For the case when the material of the cylinder is isotropic and homogeneous the classical solution of Lamé[3] is available. The radial and circumferential stress distributions are given by

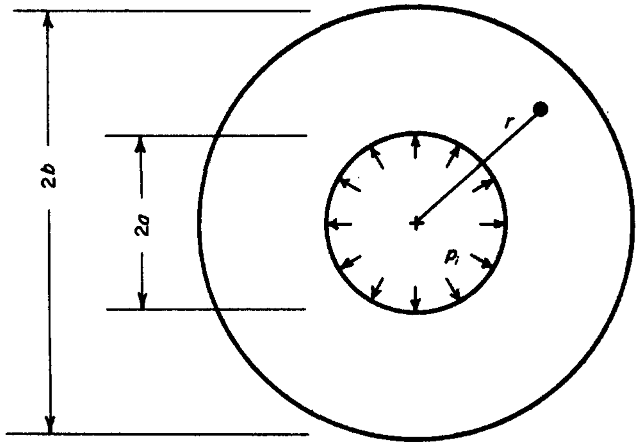

Fig. 1. Cylinder subjected to uniform pressure.

$$\sigma_r = -\frac{p_i}{k^2-1}\left(\frac{b^2}{r^2}-1\right)$$

$$\sigma_\theta = \frac{p_i}{k^2-1}\left(\frac{b^2}{r^2}+1\right). \tag{7}$$

The stress variations described by eqns (7) are shown by the solid lines in Fig. 2, wherein it is noted that

(a) $|\sigma_\theta| > |\sigma_r|$ for all r and all k.
(b) $(\sigma_\theta)_{max}$ occurs at the inner boundary ($r = a$)

whence, from eqns (7)

$$(\sigma_\theta)_{max} = \left(\frac{k^2+1}{k^2-1}\right) p_i > p_i. \tag{8}$$

The curves of Fig. 2 represent the functions given by eqns (7) for $r/b > 0.2$. It is clear that the functions became unbounded as $r/b \to 0$. Furthermore, the curves are applicable to cylinders having arbitrary ratios of boundary radii, k. Each curve is, of course, valid for the interval $1/k \leq r/b \leq 1$ for a particular problem. Thus, for a homogeneous cylinder it is seen that the

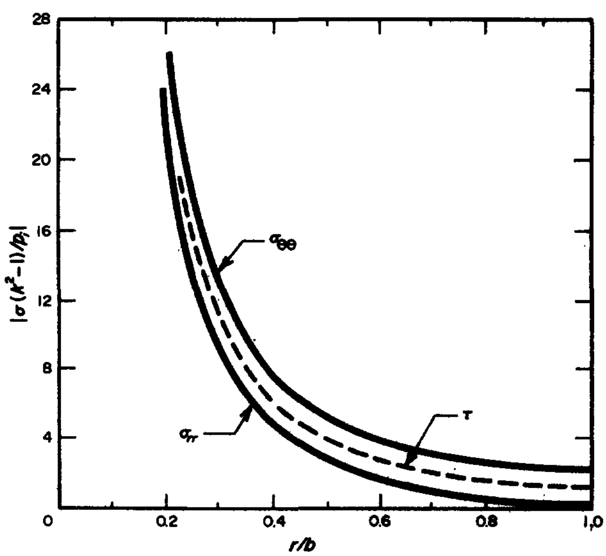

Fig. 2. Stress variation in a homogeneous isotropic pressurized cylinder.

material is not used efficiently, particularly as the ratio k increases. For example, for $k = 5$ it can be seen from Fig. 2 that the maximum of σ_θ is 13 times as large as its minimum value. From the standpoint of a maximum normal stress theory of failure, which is applicable to brittle materials, it would be desirable to have σ_θ = constant, with $\sigma_r \leq \sigma_\theta$. For ductile materials the maximum shear stress theory of failure can be applied. The maximum shear stress is given by

$$\tau_{max} = \frac{1}{2}(\sigma_\theta - \sigma_r) = \frac{p_i}{k^2 - 1}\left(\frac{b}{r}\right)^2 \tag{9}$$

which behaves as the dashed line in Fig. 2; for $k = 5$, its maximum value is 25 times its minimum.

Attempts to reduce the stress concentration in the pressurized cylinder problem were made by Bienek, Spillers and Freudenthal[6] and by Shaffer[7]. In both works nonhomogeneous, orthotropic materials were considered having elastic coefficients a_{ij} varying according to r^λ, where λ is a real constant, and optimization studies were made by observing the results. These methods were able to result in reduced stress concentration in σ_θ, but not to eliminate it. In Ref. [6] an attempt was also made to eliminate stress concentration altogether in σ_θ, by requiring σ_θ to be constant everywhere. However, that formulation of the problem was found to be intractable, and simplifying assumptions were made in order to obtain an approximate solution.

Consider now a general, nonhomogeneous, orthotropic material defined by the stress–strain eqns (4). For the present case of axisymmetry, $\gamma_{r\theta}$ is identically zero, derivatives with respect to θ vanish, and eqn (1) can be integrated to give

$$\frac{d}{dr}(r\epsilon_\theta) - \epsilon_r = 0 \tag{10}$$

for the governing equation of compatibility. For the axisymmetric problem the stresses can be most conveniently expressed in terms of a function auxiliary to the Airy stress function. That is, defining ψ by

$$\psi = \frac{d\phi}{dr} \tag{11}$$

eqns (2) can be written as

$$\sigma_r = \frac{\psi}{r} + V, \quad \sigma_\theta = \frac{d\psi}{dr} + V. \tag{12}$$

Omitting body forces and substituting eqns (4) and (12) into (10) yields

$$a_{22}\frac{d^2\psi}{dr^2} + \left(\frac{da_{22}}{dr} + \frac{a_{22}}{r}\right)\frac{d\psi}{dr} + \left(\frac{1}{r}\frac{da_{12}}{dr} - \frac{a_{11}}{r^2}\right)\psi = 0. \tag{13}$$

For a material having dependent nonhomogeneity the variable elastic coefficients a_{11} and a_{12} are related to a_{22} by eqns (6) and eqn (13) can be rewritten as

$$\left(\frac{d\psi}{dr} + c_{12}\frac{\psi}{r}\right)\frac{da_{22}}{dr} + \left(\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - c_{11}\frac{\psi}{r^2}\right)a_{22} = 0. \tag{14}$$

For a prescribed internal stress distribution, ψ is a known function of r, and eqn (14) is a first order differential equation having variable coefficients.

Let us attempt to eliminate the critical stress concentration altogether in the present problem by requiring that σ_θ be constant *everywhere* within the cylinder, say, $\sigma_\theta = A$, and then attempting to determine a suitable nonhomogeneous material.

Integrating the second of eqns (12) yields

$$\psi = Ar + B$$

$$\sigma_r = A + \frac{B}{r} \tag{15}$$

where B is a constant of integration. Applying the boundary conditions that $\sigma_r(a) = -p_i$ and $\sigma_r(b) = 0$ yields A and B, whence

$$\psi = \frac{p_i b}{k-1}\left(\frac{r}{b} - 1\right)$$

$$\sigma_r = -\frac{p_i}{k-1}\left(\frac{b}{r} - 1\right)$$

$$\sigma_\theta = \frac{p_i}{k-1}. \tag{16}$$

Comparing eqns (16) it is seen that $|\sigma_r| \leq |\sigma_\theta|$ provided that $k \leq 2$. In this case the ratio of maximum normal stresses (σ_θ) from eqns (16) and (7) is

$$\frac{\text{Homogeneous } \sigma_{\max}}{\text{Nonhomogeneous } \sigma_{\max}} = \frac{k^2+1}{k+1} \geq 1. \tag{17}$$

For $k > 2$, eqns (16) show that σ_r becomes the maximum normal stress in absolute value, but it is compressive, its value never exceeds p_i, and there is no stress concentration.

To determine a suitable, dependently nonhomogeneous material, ψ from eqns (15) is substituted into (14), yielding

$$\left(1 + c_{12} - c_{12}\frac{b}{r}\right)\frac{d\alpha_{22}}{dr} + \frac{1}{r}\left(1 - c_{11} + c_{11}\frac{b}{r}\right)\alpha_{22} = 0. \tag{18}$$

Equation (18) can be integrated exactly to yield the solution

$$\alpha_{22} = \alpha_{22}^0\left(\frac{r}{b}\right)^{-\beta}\left[c_{12} + (1 - c_{12})\left(\frac{r}{b}\right)\right]^{\xi} \tag{19}$$

where

$$\beta = \frac{c_{11}}{c_{12}}$$

$$\xi = \frac{c_{11} - c_{12}}{c_{12}(1 - c_{12})}$$

and where α_{22}^0 is a constant of integration found to be equal to the value of α_{22} at the outer boundary, $r = b$.

Examining eqns (4) and (6), it is seen that the constants c_{11} and c_{12} can be expressed in terms of the "technical" elastic coefficients of the material (E_r, E_θ, $\nu_{\theta r}$) by the relationships.

$$c_{11} = \frac{E_\theta}{E_r}, \quad c_{12} = -\nu_{\theta r} \tag{20}$$

and the results of eqn (19) are given more meaningful physical interpretation when expressed as E_θ/E_θ^0. These results are graphed in Fig. 3 for $\nu_{\theta r}$ taken to be 0.5. It is seen that eqn (19) does not depend upon k, and therefore in Fig. 3 the inner boundary may be chosen at any value of r/b. It is seen that to avoid stress concentration, E_θ must have its smallest value at the inner boundary, $r = a/b$, and increase monotonically with increasing r. For $E_\theta/E_r = 1$, the material is isotropic. Curves for E_θ/E_r equal to 20 and 100 represent possible material synthesis using glass and boron fibers, respectively. It is seen that these strongly orthotropic materials require a very high density of fibers on the outside, compared to the inside.

The case when the outer boundary, instead, is subjected to a uniform pressure, p_0, is obtained quite simply from the preceding analysis by substituting p_0 for p_i, α_{22}^i for α_{22}^0, and by interchanging a and b in eqns (16)–(19). The necessary variation in orthotropic modulus E_θ to

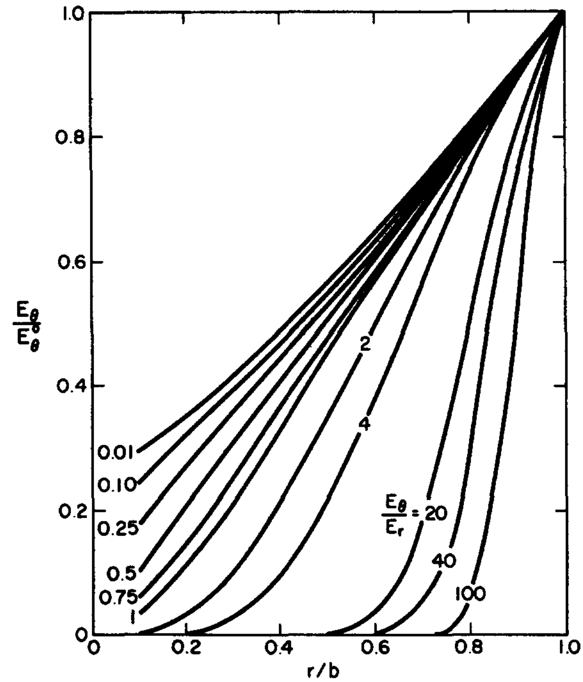

Fig. 3. Required variation in material properties to yield σ_θ = constant; *internal* pressure. E_θ/E_r variable, $\nu_{\theta r} = 0.5$.

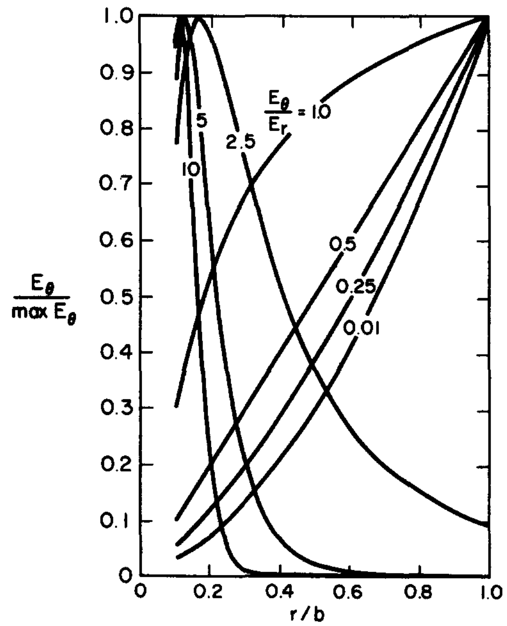

Fig. 4. Required variation in material properties to yield σ_θ = constant; *external* pressure. E_θ/E_r variable, $\nu_{\theta r} = 0.5$.

eliminate circumferential stress concentration is shown in Fig. 4 for $\nu_{\theta r} = 0.5$. The ordinate for these curves is E_θ normalized by its maximum value. It is interesting to note that the maximum E_θ may need to occur at a radius between the inner and outer radii if $E_\theta/E_r > 1$.

4. THE ROTATING DISK

Consider near the annular disk having inner and outer radii a and b, respectively, rotating about its polar axis with an angular velocity ω. Let both of its circular boundaries be free; i.e. $\sigma_r(a) = \sigma_r(b) = 0$. Internal stress and displacements are caused by the centrifugal body force of intensity $R = \rho\omega^2 r$ acting upon all internal elements. For the case of a homogeneous, isotropic material, the solution for the internal stress distribution is widely known (see [2], p. 82)

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2 a^2\left[k^2 + 1 - \left(\frac{b}{r}\right)^2 - k^2\left(\frac{r}{b}\right)^2\right]$$

$$\sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 a^2\left[k^2 + 1 + \left(\frac{b}{r}\right)^2 - \frac{1+3\nu}{3+\nu}k^2\left(\frac{r}{b}\right)^2\right]. \tag{21}$$

Examining eqns (21), one observes that $\sigma_\theta > \sigma_r$ for all r, and that the maximum and minimum values of σ_θ occur at the inner and outer boundaries, respectively. The magnitude of the variation of σ_θ with r can be seen from the ratio

$$\frac{\max \sigma_\theta}{\min \sigma_\theta} = \frac{(3+\nu)k^2 + (1-\nu)}{(1-\nu)k^2 + (3+\nu)} \tag{22}$$

which has been evaluated in Table 1 for the range of ν and k. It is seen that large stress concentration occurs at the inner boundary, especially for large k and for large ν. The value $k = \infty$ does *not* correspond to a solid disk, but rather to one having a very small hole.

The question then arises: can one eliminate stress concentrations for the rotating disk as it was done for the pressurized cylinder by choosing a nonhomogeneous material having properly varying elastic coefficients?

Requiring σ_θ to be a constant, say A, taking $V = -\rho\omega^2 r^2/2$, integrating the second of eqns (12) for ψ, and substituting the boundary conditions $\sigma_r(a) = \sigma_r(b) = 0$ yields

$$\psi = Ar + B + \frac{\rho\omega^2 r^3}{6}$$

$$\sigma_r = A + \frac{B}{r} - \frac{\rho\omega^2 r^2}{3} \tag{23}$$

$$\sigma_\theta = A$$

where

$$A = \frac{\rho\omega^2 a^2}{3}[1 + k(k+1)]$$

$$\beta = -\frac{\rho\omega^2 a^3}{3}k(k+1).$$

Limiting the analysis to dependently homogeneous materials restricted according to eqns (6) the compatibility eqn (10) is found to require that

$$(C_1 r^4 + C_2 r^2 + C_3 r)\frac{d\alpha_{22}}{dr} - (C_4 r^3 + C_5 r + C_6)\alpha_{22} = 0 \tag{24}$$

Table 1. Variation of max σ_θ/min σ_θ with radius ratio (b/a) and Poisson's ratio (ν) in a homogeneous, isotropic, rotating disk

	max σ_θ/min σ_θ *for* ν =		
$k = \frac{b}{a}$	0	0.3	0.5
1	1.00	1.00	1.00
1.5	1.48	1.61	1.80
2	1.86	2.28	2.64
3	2.33	3.17	4.00
5	2.72	4.00	5.50
10	2.92	4.52	6.55
∞	3.00	4.72	7.00

where

$$C_1 = c_{12}$$
$$C_2 = -(1 + c_{12})a^2(k^2 + k + 1)$$
$$C_3 = c_{12}a^2k(k + 1)$$
$$C_4 = c_{11} - 3c_{12}$$
$$C_5 = (1 - c_{11})(k^2 + k + 1)$$
$$C_6 = c_{11}k(k + 1).$$

Equation (24) has no readily apparent solution in closed form, but can be integrated numerically to yield α_{22}.

Results are graphed in Figs. 5 and 6 for the case when $k = b/a = 10$. The curves show the required variation in $E_\theta(= 1/\alpha_{22})$, normalized with respect to its maximum value in the disk. In Fig. 5 $\nu_{\theta r}(= -c_{12})$ is fixed at 0.5 and $E_\theta/E_r(= c_{11})$ is permitted to vary. In Fig. 6 E_θ/E_r is fixed at 2.5 and $\nu_{\theta r}$ is varied.

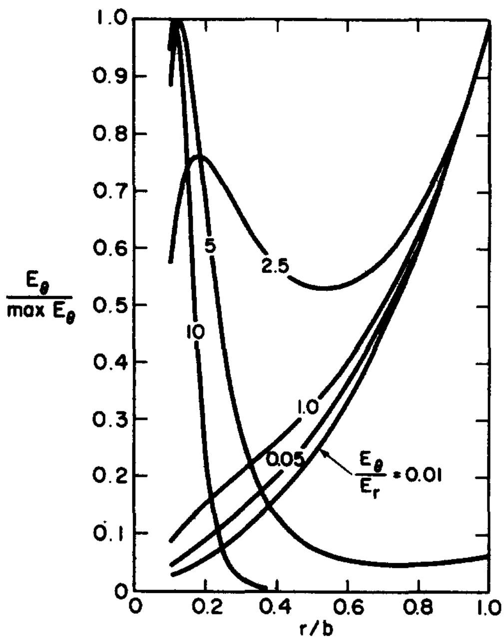

Fig. 5. Required variation in material properties to yield σ_θ = contant; rotating disk, E_θ/E_r variable, $\nu_{\theta r} = 0.5$, $b/a = 10$.

5. SOME FURTHER OBSERVATIONS

The preceding examples for the pressurized cylinder and rotating disk were solved independently of each other. Clearly, a more general solution can be found which permits the simultaneous consideration of internal and external pressure and angular velocity, yielding modulus variations which eliminate stress concentration.

The preceding examples have been subjected to the criterion that the maximum normal stress be constant, which is usually applied to brittle materials. For a ductile material, a maximum shear stress criterion of failure might be applied. Because σ_r and σ_θ are principal stresses for axisymmetric problems, the maximum shear stress is simply

$$\tau_{max} = \frac{1}{2}(\sigma_\theta - \sigma_r). \tag{25}$$

Requiring τ_{max} to be a constant everywhere for the cylinder having internal pressure (p_i), the equations of equilibrium are found to require that

$$\sigma_r = -p_i\frac{\ln(b/r)}{\ln k}, \quad \sigma_\theta = p_i\frac{1 - \ln(b/r)}{\ln k}. \tag{26}$$

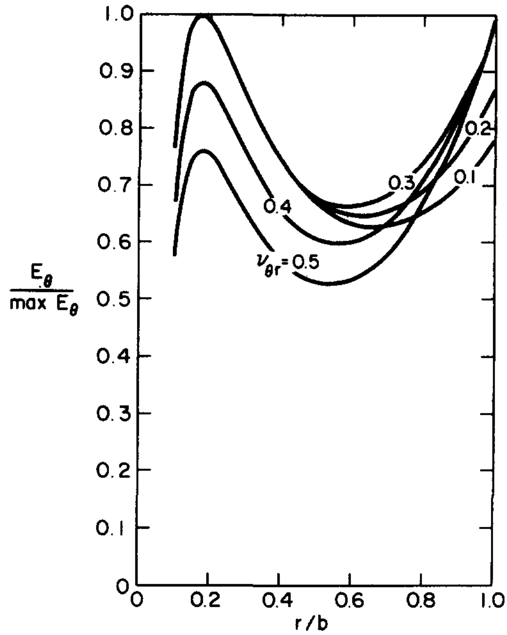

Fig. 6. Required variation in material properties to yield σ_θ = constant; rotating disk, $E_\theta/E_r = 2.5$, $\nu_{\theta r}$ variable, $b/a = 10$.

For the simple case of a nonhomogeneous, isotropic material the compatibility eqns (10) can be integrated to yield

$$E = E^0\left[(1-\nu)\ln\frac{r}{b}+1\right]^{2/(1-\nu)} \tag{27}$$

for ν = constant. The inversion of boundaries to yield the solution for external pressure, as before, is a simple matter.

For many materials the elastic moduli and failure stresses are both approximately proportional to the density of the material. In such cases it may be desirable to consider maximum *strain*, rather than stress, as the criterion to be optimized.

Finally, the examples included in this work can be handled relatively simply because they are axisymmetric (i.e. there is no variation with θ). Returning to the problem of a flat panel with a small hole subjected to a uniaxial tensile stress, which was discussed in the Introduction, one finds that a possible solution for elastic moduli to optimize stress would involve *all* the Fourier components $\cos 2n\theta$ (i.e. $n = 0, 1, 2, \ldots \infty$), leading to the simultaneous solution of an infinite set of coupled differential equations.

6. CONCLUSIONS

In the preceding work it was shown that it is possible to determine variations in elastic moduli to reduce, and even eliminate, undesirable concentrations of stress. Many materials are available which are capable of exhibiting significant variations in moduli (e.g. foams, elastomers, polymers, fibrous composites) and, therefore, stress levels can be controlled if the technology of controlling elastic moduli is developed.

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